Turing Machines and Undecidability

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Motivation

- Undecidability has been regarded as one of the most philosophically important problems in the theory of computation
- If your problem is undecidable, good luck deciding

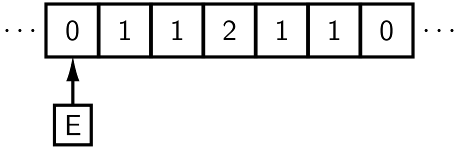
Turing Machine

Definition

A **Turing machine** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are finite sets and

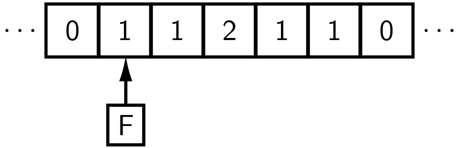
1. Q is the set of states
2. Σ is the input alphabet not containing the **blank symbol** $\sqcup$
3. Γ is the tape alphabet (symbols), where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$
4. $\delta \colon Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function
5. $\{q_0, q_{\text{accept}}, q_{\text{reject}}\} \subseteq Q$ and are the start state, accept state, and reject state respectively where $q_{\text{accept}} \neq q_{\text{reject}}$

An Example



State \ Symbol	0	1	2
A	✓,0,R	E,0,L	×,2,L
B	✓,0,L	×,1,L	F,0,L
C	A,0,L	C,1,R	C,2,R
D	B,0,L	D,1,R	D,2,R
E	F,0,R	E,1,L	E,2,L
F	✓,0,R	C,0,R	D,0,R

An Example



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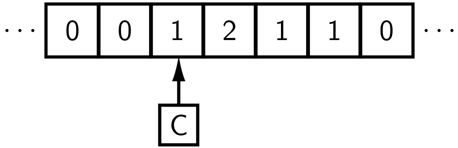
An Example



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D	B,0,L	D,1,R	D,2,R
E	F,0,R	E,1,L	E,2,L
F	✓,0,R	C,0,R	D,0,R

Turing Recognizable and Decidable

Definition (Turing-recognizable)

A language is **Turing-recognizable** if some Turing machine recognizes it.

Definition (Turing-decidable)

A language is **Turing-decidable** or just **decidable** if some Turing machine decides it.

A New Language

$$A_{\mathsf{TM}} := \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

Theorem

A_{TM} is Turing-recognizable.

Proof.

$U =$ "On input $\langle M, w\rangle$ where M is a TM and w is a string:

1. Simulate M on w
2. If M ever enters an accept state, *accept*; if M ever enters a reject state, *reject*"

□

Is A_{TM} decidable?

Let us assume A_{TM} is decidable. Then there exists a TM H such that

$$H(\langle M, w\rangle) := \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Lets also define a new TM D such that
$D =$ "On input $\langle M \rangle$ where M is a Turing machine:

1. Run H on input $\langle M, \langle M \rangle \rangle$
2. Output the opposite of H"

What does H look like?

M \ w	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*		*accept*	
M_2	*reject*	*accept*	*reject*	$\cdots$	*reject*	$\cdots$
M_3	*accept*	*accept*	*reject*		*accept*	
$\vdots$		$\vdots$		$\ddots$		
D	*accept*	*reject*	*reject*		$\underline{?}$	
$\vdots$		$\vdots$				$\ddots$

What happens at ?

$$D(\langle D \rangle) := \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D \rangle \end{cases}$$

Contradiction!
What ever D does, it is forced to do the opposite. Therefore, our assumption if false and A_{TM} is not decidable.

Interesting Results

- Kurt Gödel showed there are undecidable problems in number theory
- Conway showed a generalization of the famous Collatz problem is undecidable
- Hilbert's 10th problem
- and in general, any nontrivial property about the language recognized by a Turing machine is undecidable (Rice's theorem)

Sources

1. Sipser, M. (2013). Introduction to the Theory of Computation. Boston, MA: Course Technology. ISBN: 113318779X
2. John H. Conway. (2013). On Unsettleable Arithmetical Problems. The American Mathematical Monthly, 120(3), 192–198. https://doi.org/10.4169/amer.math.monthly.120.03.192

Thank You!